Response of Discrete Linear Systems to Forcing Functions with Inequality Constraints

C. D. MICHALOPOULOS* AND T. A. RILEY†
University of Houston, Houston, Texas

An analysis is made of the maximum response of discrete, linear mechanical systems to arbitrary forcing functions which lie within specified bounds. Primary attention is focused on the complete determination of the forcing function which will engender maximum displacement to any particular mass element of a multi-degree-of-freedom system. In general, the desired forcing function is found to be a bang-bang type function, i.e., a function which switches from the maximum to the minimum bound and vice-versa at certain instants of time. Examples of two-degree-of-freedom systems, with and without damping, are presented in detail. Conclusions are drawn concerning the effect of damping on the switching times and the general procedure for finding these times is discussed.

Introduction

IN certain applications of the field of structural dynamics, it may be desirable to estimate the maximum response of a multi-degrees-of-freedom system to forcing functions whose dependence on time is not precisely known. It is possible, for instance, to know the limiting values of the forcing function and other conditions of constraint such as initial values but not the function itself. In this situation it may be necessary to determine which excitation produces maximum strain in specified members of the structure. If this is determined it is possible to design the structure such that the allowable strains are not exceeded whatever the excitation forces may be, as long as they are within the specified limits.

Problems with bounded control variables (forcing functions) are found extensively in the field of optimal control.[1–4] This paper presents an application of optimal control techniques to the structural-dynamics discipline. Much effort has been directed toward the solution of control problems whose control variables are bounded in absolute value. The problem of investigating the optimal forcing function when the bounds change with time is also of considerable interest. The linear dependence of a bounded control variable in both the equations of motion and the performance index results in a "bang-bang" optimal control problem.[2]

Some examples where only bounds of the forcing function (excitation) may be known, but not the function itself, are: a) thrust decay or thrust start-up of a rocket, b) landing of aircraft, particularly on unprepared surfaces, c) discrete wind-gust loadings on aircraft, and d) landing of a lunar module. In such cases, determination of the excitation which induces maximum relative displacement of adjacent masses may be important to the design of specified members of a structure if the dynamic loads are not to be exceeded.

The object of this investigation is to determine the excitation, subject to certain inequality constraints, which when applied at a given point of a discrete, linear structure will induce maximum response of any specified mass element.

Problem Statement

Consider a discrete, linear, mechanical system having n degrees of freedom whose motion is governed by the differential equations

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{Q(t)\} \qquad (1$$

where $[M]$, $[C]$, and $[K]$ are the inertia, damping, and stiffnes matrices, respectively; $\{q\}$ is the vector of generalized coo dinates; and $\{Q\}$ is the vector of generalized forces. For simpl city, only one component of $\{Q\}$, say the jth, is taken to b nonzero. This unknown function of time $Q_j(t)$ is required t satisfy the inequality constraint

$$P^l(t) \leqq Q_j(t) \leqq P^u(t) \qquad (2$$

where $P^l(t)$ and $P^u(t)$ are specified. For the special case of monotonically decreasing function from an initial value Q_0, th required function $Q_j(t)$ may be one of the functions shown b the dotted lines in Fig. 1.

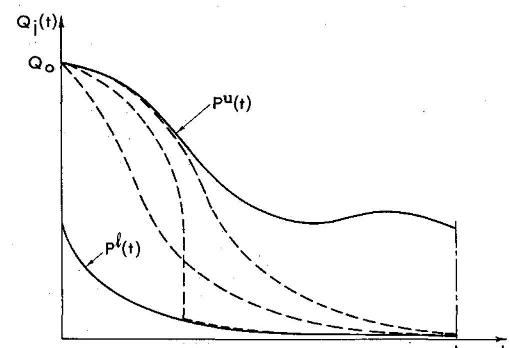

Fig. 1 Upper and lower bounds of forcing function.

The problem to be investigated can now be stated. Find th forcing function $Q_j(t)$ satisfying the inequality constraint (2 which will maximize a specified component of $\{q\}$—say $q_i(t$ for $t \leqq t_m$ (t_m being the time up to which $P^l(t)$ and $P^u(t)$ ar specified). The initial conditions are taken to be zero.

Optimal Control Formulation

In state space, Eqs. (1) are equivalent to $2n$ first-order equa tions which can be written as

$$\{\dot{x}(t)\} = [A]\{x(t)\} + [B]\{F(t)\} \qquad (3$$

where $\{x\}$ is the vector of state variables whose components ar defined by

$$x_{2i-1} = q_i, \quad x_{2i} = \dot{q}_i, \quad i = 1, 2, \ldots, n \qquad (4$$

In functional notation, Eqs. (3) may be written as

$$\dot{\bar{x}} = \bar{R}(\bar{x}, \bar{F}, t) \qquad (5$$

where $\bar{x} \equiv \{x\}$.

If U denotes the set of all functions such that

$$P^l(t) \leqq Q_j(t) \leqq P^u(t) \qquad (6$$

Received November 22, 1971; revision received April 21, 1972. The authors are deeply indebted to NASA for financial support of this study under Grant NGR 44-005-120. The authors are also grateful to C. Glynn, Chief, Structures Branch, Structures and Mechanics Division, NASA Manned Spacecraft Center, Houston, Texas, for his suggestions.

Index category: Structural Dynamic Analysis.

* Associate Professor of Mechanical Engineering.

† Assistant Professor of Mechanical Engineering.

the problem reduces to determining the optimal $Q_j(t) \in U$, say $Q_j^*(t)$, such that a specified coordinate $q_i = x_{2i-1}$ is a maximum. The maximization of q_i is equivalent to minimizing its negative; thus, it is sufficient to minimize the performance index

$$J = -x_{2i-1} \tag{7}$$

subject to Eqs. (5) and (6) and the initial conditions

$$\bar{x}(0) = 0 \tag{8}$$

The Hamiltonian

$$H = \bar{\lambda}^T \bar{R} \tag{9}$$

may now be formed, $\bar{\lambda}(t)$ being a vector of Lagrange multipliers. If optimal conditions are designated with an asterisk, the necessary conditions for any state variable to be a minimum are the state equations

$$\dot{x}_i^* = \partial H / \partial \lambda_i; \qquad i = 1, 2, \ldots, 2n \tag{10}$$

and the costate equations

$$\dot{\lambda}_i^* = -\partial H / \partial x_i \tag{11}$$

which are to be applied concurrently with Pontryagin's minimum principle,[4] viz.

$$H(\bar{x}^*, \bar{F}^*, \bar{\lambda}^*, t) \leqq H(\bar{x}^*, \bar{F}, \bar{\lambda}^*, t) \tag{12}$$

Since $\bar{x}(t_f)$ and t_f are free, necessary end conditions are[4]

$$\lambda_i^*(t_f) - \frac{\partial J}{\partial x_i}\bigg|_{t_f} = 0, \quad i = 1, 2, \ldots, 2n \tag{13}$$

$$H[\bar{x}^*(t_f), \bar{F}^*(t_f), \bar{\lambda}^*(t_f), t_f], t_f] + \frac{\partial J}{\partial t}\bigg|_{t_f} = 0 \tag{14}$$

Equations (8–14) complete formulation of the problem in state space for a linear system with n degrees of freedom.

A System with Two Degrees of Freedom

To keep the problem tractable, the equations of the preceding section are solved in this section for a system with two degrees of freedom. It should be noted that such a system possesses the basic characteristics of any n-degree-of-freedom system; an understanding of the behavior of a general, linear, discrete system can be obtained by analyzing a system with two degrees of freedom. The system under consideration is shown in Fig. 2.

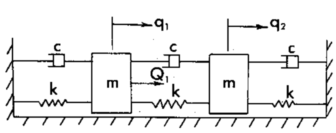

Fig. 2 System analyzed in detail.

The differential equations of motion for this system are

$$\begin{aligned} m\ddot{q}_1 + 2c\dot{q}_1 - c\dot{q}_2 + 2kq_1 - kq_2 &= Q_1 \\ m\ddot{q}_2 + 2c\dot{q}_2 - c\dot{q}_1 + 2kq_2 - kq_1 &= 0 \end{aligned} \tag{15}$$

Equations (3) in this case are

$$\begin{aligned} \dot{x}_1 = x_2, \quad \dot{x}_2 &= -2bx_2 + bx_4 - 2\omega^2 x_1 + \omega^2 x_3 + aQ_1 \\ \dot{x}_3 = x_4, \quad \dot{x}_4 &= -2bx_4 + bx_2 - 2\omega^2 x_3 + \omega^2 x_1 \end{aligned} \tag{16}$$

where

$$a = 1/m, b = c/m, \text{ and } \omega^2 = k/m$$

The Hamiltonian function for the system is

$$\begin{aligned} H = \lambda_1 x_2 + \lambda_2(aQ_1 - 2bx_2 + bx_4 - 2\omega^2 x_1 + \omega^2 x_3) \\ + \lambda_3 x_4 + \lambda_4(-2bx_4 + bx_2 - 2\omega^2 x_3 + \omega^2 x_1) \end{aligned} \tag{17}$$

Thus, the costate equations, Eqs. (11), are

$$\begin{aligned} \dot{\lambda}_1 = 2\omega^2\lambda_2 - \omega^2\lambda_4, \quad \dot{\lambda}_2 = -\lambda_1 + 2b\lambda_2 - b\lambda_4 \\ \dot{\lambda}_3 = -\omega^2\lambda_2 + 2\omega^2\lambda_4, \quad \dot{\lambda}_4 = -b\lambda_2 - \lambda_3 + 2b\lambda_4 \end{aligned} \tag{18}$$

Pontryagin's Maximum Principle requires that

$$\lambda_2 Q_1^*(t) \leqq \lambda_2 Q_1(t) \tag{19}$$

Thus, the optimal $Q_1(t) \in U$ is given by

$$Q_1^*(t) = P^u(t) \quad \text{if} \quad \lambda_2 < 0 \tag{20a}$$

$$Q_1^*(t) = P^l(t) \quad \text{if} \quad \lambda_2 > 0 \tag{20b}$$

If the displacement x_1 is to be maximized (displacement of mass on which the forcing function is applied), the performance index is

$$J = -x_1(t_f) \tag{21}$$

Application of Eqs. (13) yields the following boundary values for the Lagrange multipliers:

$$\left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \\ \lambda_4 &= 0 \end{aligned}\right\} \text{ at } \quad t = t_f \tag{22}$$

Equations (18) may be written in the form

$$\dot{\bar{\lambda}} = G\bar{\lambda} \tag{23}$$

The solution of Eq. (23) is

$$\bar{\lambda} = C_1 e^{\Lambda_1 t}\bar{e}_1 + C_2 e^{\Lambda_2 t}\bar{e}_2 + C_3^{\Lambda_3 t}\bar{e}_3 + C_4 e^{\Lambda_4 t}\bar{e}_4 \tag{24}$$

where Λ_i is an eigenvalue of G and $\bar{e}_i$ is the associated eigenvector. The matrix G of Eq. (23) is

$$G = \begin{bmatrix} 0 & 2\omega^2 & 0 & -\omega^2 \\ -1 & 2b & 0 & -b \\ 0 & -\omega^2 & 0 & 2\omega^2 \\ 0 & -b & -1 & 2b \end{bmatrix}$$

its eigenvalues being

$$\Lambda_{1,2} = [b \pm (b^2 - 4\omega^2)^{1/2}]/2 \tag{25a}$$

$$\Lambda_{3,4} = [3b \pm (9b^2 - 12\omega^2)^{1/2}]/2 \tag{25b}$$

A. Undamped System

If the damping is zero, i.e., $b = 0$, the eigenvalues become

$$\Lambda_{1,2} = \pm i\omega = \pm i\omega_1, \quad \Lambda_{3,4} = \pm i\omega(3)^{1/2} = \pm i\omega_2 \tag{26}$$

where $\omega_1 = (k/m)^{1/2}$, $\omega_2 = (3k/m)^{1/2}$ are the undamped natural frequencies of the system.

It can be shown that the ith eigenvector corresponding to Λ_i of Eq. (26) is

$$\bar{e}_i = \begin{Bmatrix} -\Lambda_i \\ 1 \\ -\Lambda_i(\Lambda_i^2/\omega_1^2 + 2) \\ \Lambda_i^2/\omega_1^2 + 2 \end{Bmatrix} \tag{27}$$

Usings Eqs. (27) and (26) in (24) and applying conditions (22) results in the following solution for λ_2:

$$\lambda_2(t) = (1/2\omega_1)\sin\omega_1(t - t_f) + (1/2\omega_2)\sin\omega_2(t - t_f) \tag{28}$$

The zeros of $\lambda_2(t)$ are roots of the transcendental equation

$$\omega_2 \sin\omega_1(t_f - t) + \omega_1 \sin\omega_2(t_f - t) = 0 \tag{29}$$

The times at which $\lambda_2(t)$ changes sign are from Eqs. (20), precisely the times at which the optimal forcing function jumps from its maximum to its minimum bound or vice versa. Equation (29) shows that these switching times are not periodic in general, unless ω_2 is an integral multiple of ω_1. For the particular example considered, $\omega_2 = (3)^{1/2}\omega_1$.

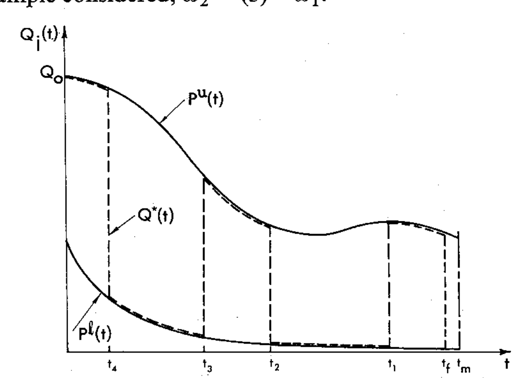

Fig. 3 Optimal forcing function for system of Fig. 2.

The first three roots of $\lambda_2(t_f - t)$ are approximately

$$t_1 = t_f - 2.55/\omega_1, \quad t_2 = t_f - 6.66/\omega_1, \quad t_3 = t_f - 9.33/\omega_1$$

It should be noted that $0 < \ldots < t_3 < t_2 < t_1 < t_f \leqq t_m$ where t_m is specified. Since $\lambda_2(t)$ is negative for $t = 0$, the optimal forcing function, shown qualitatively in Fig. 3, starts with its maximum. It has the value $P^u(t)$ up to $t = t_k$ at which time it jumps to its lower bound, $P^l(t)$, continues at the lower bound until $t = t_{k-1}$, where it jumps to the upper bound, $P^u(t)$, and so on. Since t_m is given, the number of switching times can easily be determined for a particular problem.

B. Damped System

In familiar notation, the expressions for the eigenvalues of the damped system in question, Eqs. (25), become

$$\Lambda_{1,2} = \zeta_1\omega_1 \pm i\omega_{d_1}, \quad \lambda_{3,4} = \zeta_2\omega_2 \pm i\omega_{d_2} \tag{30}$$

where ζ_1 and ζ_2 are the damping ratios for the first and the second mode, respectively, and ω_{d_1} and ω_{d_2} are the damped natural frequencies. It should be noted that for this system $\zeta_1 = c/2m\omega_1$, $\zeta_2 = (3)^{1/2}\zeta_1$ and $\omega_{d_i} = \omega_i(1 - \zeta_i^2)^{1/2}$. The system is assumed "underdamped," i.e., $\zeta_1, \zeta_2 < 1$. It can be shown that the eigenvectors in this case are

$$\bar{e}_1 = \begin{Bmatrix} \Lambda_2 \\ 1 \\ \Lambda_2 \\ 1 \end{Bmatrix}, \quad \bar{e}_2 = \begin{Bmatrix} \Lambda_1 \\ 1 \\ \Lambda_1 \\ 1 \end{Bmatrix}, \quad \bar{e}_3 = \begin{Bmatrix} -\Lambda_4 \\ -1 \\ \Lambda_4 \\ 1 \end{Bmatrix}, \quad \bar{e}_4 = \begin{Bmatrix} -\Lambda_3 \\ -1 \\ \Lambda_3 \\ 1 \end{Bmatrix} \tag{31}$$

Substituting Eqs. (31) and (30) in (24) and applying conditions (22) yields for the Lagrange multiplier λ_2

$$\lambda_2(t) = (1/2\omega_{d_1})\exp[-\zeta_1\omega_1(t_f - t)]\sin\omega_{d_1}(t - t_f) + (1/2\omega_{d_2})\exp[-\zeta_2\omega_2(t_f - t)]\sin\omega_{d_2}(t - t_f) \tag{32}$$

The zeros of $\lambda_2(t)$ are roots of the transcendental equation

$$\sin\omega_{d_1}(t_f - t) + (\omega_{d_2}/\omega_{d_1})\exp[-(\zeta_2\omega_2 - \zeta_1\omega_1)(t_f - t)]\sin\omega_{d_2}(t_f - t) = 0 \tag{33}$$

Since $\zeta_2\omega_2 > \zeta_1\omega_1$, the second term becomes negligible after some interval $(t_f - t)$, due to the presence of the exponential factor. This, of course, implies that, for reasonable damping, the roots of (33) become very nearly periodic after the first few. Thus, for a damped system it is easier to determine the switching times of the optimal forcing function than it is for the same undamped system. It should be recalled that the roots of Eq. (29) are not periodic unless ω_1 and ω_2 are commensurable. As an example, consider the roots of Eq. (33) for $c = 1$, $k = 2$, $m = 1$. For these values, $\omega_1 = (2)^{1/2}$, $\omega_2 = (6)^{1/2}$, $\zeta_1\omega_1 = \frac{1}{2}$, $\zeta_2\omega_2 = \frac{3}{2}$, $\omega_{d_1} = (7)^{1/2}/2$, $\omega_{d_2} = (15)^{1/2}/2$. Eq. (33) becomes

$$\sin[(7)^{1/2}/2](t_f - t) + (15/7)^{1/2}\exp[-(t_f - t)]\sin[(15)^{1/2}/2](t_f - t) = 0 \tag{34}$$

As previously discussed, the roots to this equation become periodic, with period $4\pi/(7)^{1/2}$, once the exponential term eliminates the contribution from ω_{d_2}, which is accomplished almost immediately in this particular example. The presence of damping in the problem tends to reduce the effort involved in obtaining the switching times.

C. Further Remarks on Two-Degree-of-Freedom Systems

In both examples, the displacement of the mass to which the forcing function is applied has been of primary concern, and a general procedure for obtaining the switching times has been outlined. In general, one may wish to apply the forcing function to one mass and maximize the displacement of some other mass. For the two-degree-of-freedom example considered, the performance index becomes $J = -x_3(t_f)$, while all other conditions remain unchanged. Herein lies the advantage of choosing the performance index as a final condition, rather than as an integral condition. The Hamiltonian is invariant for any displacement which is to be maximized; thus, for any specific problem, the general solution to only one set of costate equations must be found. Once it is obtained, the particular solution sought is found by evaluating the constants for the appropriate end conditions.

Thus far, it has been taken for granted that the optimal control switches instantaneously between its upper and lower bounds with a sign change in the pertinent Lagrange multiplier. Physically this seems reasonable, but the possibility of a singular solution must be investigated.

To pursue this matter, one assumes a priori knowledge of a finite time interval during which the Lagrange multiplier identically vanishes. For the discrete linear systems involved, such an assumption yields only the trivial solution to the costate equations, which prevents one from satisfying the conditions at the final time. Thus, no singular (bang-zero-bang) solution exists.

Determination of the Final Time

The solution of the differential equations of motion, Eqs. (1), or the state equations, Eqs. (10), is[5]

$$\{q(t)\} = \sum_{r=1}^{n} \frac{\{\Phi\}_r\{\Phi\}_r^T}{\omega_{rd}} \int_0^t \{Q(\tau)\}e^{-\zeta_r\omega_r(t-\tau)}\sin\omega_{rd}(t - \tau)\,d\tau \tag{35}$$

where ω_r = rth natural frequency, $\{\Phi\}_r$ = rth modal column, $\zeta_r = \{\Phi\}_r^T[c]\{\Phi\}_r/2\omega_r$ (damping ratio of the rth mode), and $\omega_{rd} = (1 - \zeta_r^2)^{1/2}\omega_r$ (rth damped natural frequency). Solution (35) assumes the existence of classical normal modes in the damped system.[6] Let $q_i(t)$ be the coordinate to be maximized and $Q_j(t)$ be the only nonvanishing component of the excitation. If $t_1, t_2, \ldots, t_k$ are the switching times, i.e., the times at which the forcing function jumps from the upper to the lower bound or vice versa, then, from Eq. (35)

$$q_i(t_f) = \sum_{r=1}^{n} \frac{\Phi_{ir}\Phi_{jr}}{\omega_{rd}} \left\{ \int_0^{t_k} P_j^u(\tau)e^{-\zeta_r\omega_r(t_f-\tau)}\sin\omega_{rd}(t_f - \tau)d\tau + \int_{t_k}^{t_{k-1}} P_j^l(\tau)e^{-\zeta_r\omega_r(t_f-\tau)}\sin\omega_{rd}(t_f - \tau)d\tau + \ldots + \int_{t_1}^{t_f} P_j^u(\tau)e^{-\zeta_r\omega_r(t_f-\tau)}\sin\omega_{rd}(t_f - \tau)d\tau \right\} \tag{36}$$

where P_j^u and P_j^l are the upper and lower bounds of Q_j, respectively, and $t_k < t_{k-1} < \ldots < t_1$ are all functions of t_f as determined from the solution of Eq. (34).

Differentiating Eq. (36) and equating the right-hand side to zero yields the desired value of t_f. The switching times can then easily be obtained and the maximum value of q_i can be determined by evaluating the response integrals of Eq. (36).

Summary and Conclusions

An analysis similar to that of the fourth section applied to a system with one degree of freedom shows that the switching times $t_k, t_{k-1}, \ldots, t_1$ of the forcing function are periodic and given by

$$t_k = t_f - k\pi/(1 - \zeta^2)^{1/2}\omega_n, \quad k = 0, 1, 2, \ldots; \quad \zeta < 1 \tag{37}$$

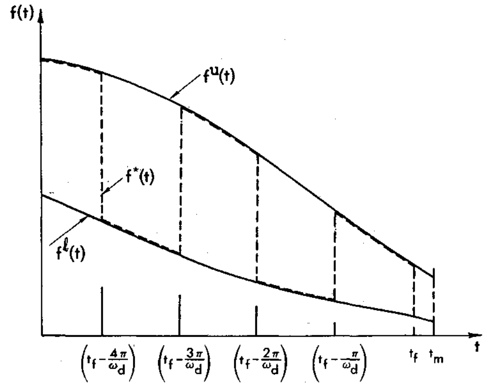

Fig. 4 Optimal forcing function for a system with one degree of freedom.

where ω_n is the undamped natural frequency of the system and ζ is the damping ratio. The optimal forcing function for one-degree-of-freedom systems is sketched in Fig. 4, in which only four switching times are shown (this implies that t_m, interval of interest, was specified as being between two and three natural periods).

It is observed from Eq. (36), or the analogous equation for the system with one degree of freedom, that t_f does in general depend on the upper and lower-bound functions. On the other hand, it has been found that all switching times except for the first and the last one (t_f) do not depend on the nature of the excitation bounds. It can easily be shown, however, that if the bounds are constant, the final time as well as the first switching time are independent of the constant values of the two bounds. In fact, for one-degree-of-freedom systems, if the two bounds are constant, $t_k = \pi/\omega_d$, $t_{k-1} = 2\pi/\omega_d$ and so on. This last observation can be deduced by intuition.

If the additional constraint is imposed that the desired forcing function is monotonically decreasing, the solution can also be obtained by applying classical calculus of variations to the response integral. In such case, the Valentine Method[7] can be used to convert the inequality constraints on the forcing function to equality constraints. Assuming that the upper and lower bounds are decreasing functions, the solution of the problem is a forcing function which is equal to the upper bound until a time t_0 where it jumps to the lower bound. For $t > t_0$, the optimal function equals to the lower bound.

This study demonstrates the applicability of optimal control theory to structural dynamics problems. Although the efforts herein utilize this approach only for determining the optimal forcing function required to produce maximum displacements in discrete linear systems, the same method may be used to determine the forcing function which maximizes a) the relative displacement of any two adjacent masses and b) the acceleration of a given mass. Relative displacement is proportional to stress in the member connecting two masses while accelerations are proportional to the dynamic loads.

Once the forcing function is determined, the system response can be obtained by conventional structural dynamics methods, such as modal analysis or the direct-integration method. Hence, the results of this study yield the input to programs which compute system response to specific forcing functions.

Finally, it should be mentioned that any complex structure may be discretized by a number of well-known accurate methods (finite elements, finite differences). In fact, a complex structure is practically always treated as a discrete system. This implies that extension of this study to continuous systems is not important from an applications point of view.

References

[1] Drefus, S. E., "Variational Problems with Inequality Constraints," *Journal of Mathematical Analysis and Applications*, Vol. 4, No. 2, April 1962, pp. 297–308.

[2] Sage, A. P., *Optimum Systems Control*, Prentice-Hall, Englewood Cliffs, N. J., 1968.

[3] Petrov, I. P., *Variational Methods in Optimal Control Theory*, Academic Press, New York, 1968.

[4] Kirk, D. E., *Optimal Control Theory*, Prentice-Hall, Englewood Cliffs, N. J., 1970.

[5] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, New York, 1967.

[6] Caughey, T. K., "Classical Normal Modes in Damped Linear Systems," *Journal of Applied Mechanics*, Vol. 27, No. 2, June 1960, pp. 269–271.

[7] Valentine, F. A., "The Problem of Lagrange with Differential Inequalities as Added Side Constraints," *Contributions to the Calculus of Variations 1933-1937*, The University of Chicago Press, 1937.